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Semi-Classical Spectrum of the Homogeneous sine-Gordon theories

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ABSTRACT

The semi-classical spectrum of the Homogeneous sine-Gordon theories associated with an arbitrary compact simple Lie group G is obtained and shown to be entirely given by solitons. These theories describe quantum integrable massive perturbations of Gepner's G-parafermions whose classical equations-of-motion are non-abelian affine Toda equations. One-soliton solutions are constructed by embeddings of the $SU(2)$ complex sine-Gordon soliton in the regular $SU(2)$ subgroups of G. The resulting spectrum exhibits both stable and unstable particles, which is a peculiar feature shared with the spectrum of monopoles and dyons in $N = 2$ and $N = 4$ supersymmetric gauge theories.

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1. Introduction

The idea that extended particles in quantum field theory can be associated with soliton solutions of the classical equations-of-motion dates back to the early work of Skyrme [1]. Well known examples of this are provided by the Skyrme model, where solitons are identified with the baryons of large-N_c QCD [2], and by monopoles and dyons, which arise as solitons in non-abelian gauge theories [3]. In most cases, the spectrum and interactions of this kind of particles can only be found at weak coupling limit, and it is extremely useful to have simple solvable models that capture as many features of realistic solitons as possible. In this sense, integrability has proved to be a powerful tool and, in fact, an important paradigm for Skyrme's ideas is provided by the sine-Gordon (SG) field theory, where an exact description of the soliton dynamics can be deduced [4], which matches precisely with semi-classical approaches in the appropriate limit. Results like this support our confidence that similar semi-classical approaches are equally valid in more realistic theories in higher dimensions.

Many integrable generalizations of the SG equations-of-motion have been written down and are known as the (abelian and non-abelian) affine Toda equations. However, not all these equations can be understood as the equations-of-motion of an action with sensible properties like a positive-definite kinetic term and a real potential. Actually, it has been shown in [5] that these requirements, together with the existence of soliton solutions, select only two series of affine Toda equations that give rise to theories with a mass-gap and, hence, that are expected to admit an S-matrix description. In that article, these two series of theories are referred to as Homogeneous sine-Gordon (HSG) theories and Symmetric Space sine-Gordon (SSSG) theories, which are associated with compact simple Lie groups and compact symmetric spaces, respectively.

The HSG and SSSG theories describe integrable perturbations of $c > 1$ conformal field theories which, in the classical limit, are formulated by means of a unitary Lagrangian. In contrast, recall the case of the abelian affine Toda field theories. There, both their interpretation as integrable perturbations of the ($c < 1$) minimal models [6] and the existence of soliton solutions [7] require an imaginary coupling constant, which, unless for the sine-Gordon theory, renders their potential complex [8]. An important consequence of having a proper unitary Lagrangian description is the possibility of addressing the quantum properties of the HSG and SSSG theories by using both the standard methods of quan-

tum field theory, and the techniques originally designed by Zamolodchikov for studying integrability [9].

The purpose of this paper is to study the classical and semi-classical soliton spectrum of the HSG theories, whose quantum integrability has been established in [10]. The construction and main features of these theories are briefly summarized in Section 2. Their classical action consists of the gauged Wess-Zumino-Witten (WZW) action corresponding to the $G/U(1)^{\times r_g}$ coset model perturbed by a potential, where G is a compact simple Lie group of rank r_g. This action exhibits a global $U(1)^{\times r_g}$ symmetry and, hence, solitons carry a conserved (Noether) vector charge $\boldsymbol{Q}$. At the quantum level, the HSG theories describe massive perturbations of the the theories of G-parafermions [11,12] where the potential induces the complete breaking of the conformal symmetry. The simplest HSG theory, associated with $G = SU(2)$, corresponds to the perturbation of the $\mathbb{Z}_k$-parafermions by the first thermal operator, and its equation-of-motion is the complex sine-Gordon (CSG) equation [13,14].

Out of the different methods available to construct explicit soliton solutions of non-abelian affine Toda equations, we will use the so-called 'solitonic specialization' proposed by Olive *et al.* [15,16], whose relation with the method of dressing transformations and τ-functions has been recently clarified in [17]. The application of this method to the HSG theories is discussed in Section 3, and we use it in Section 4 to obtain the one-soliton solutions of the theories associated with simply-laced Lie groups. In this case, explicit solutions are obtained by means of the homogeneous vertex operator construction of the untwisted affine Kac-Moody algebra associated with g. The main result of this section is that there is a one-soliton solution associated with each root $\boldsymbol{\alpha}$ of g that is nothing else but a CSG soliton embedded in the regular $SU(2)$ subgroup of G generated by $E_{\boldsymbol{\alpha}}$, $E_{-\boldsymbol{\alpha}}$, and $[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}]$.

This result is exploited in Section 5 to construct the one-soliton solutions of the HSG theories associated with arbitrary compact simple Lie groups by embedding the CSG soliton in the regular $SU(2)$ subgroups of G, which are labelled by the roots of g. Actually, this method is widely used in the context of Yang-Mills theories based on arbitrary Lie groups to construct monopole or instanton solutions by embeddings of the $SU(2)$ spherically symmetric 't-Hooft-Polyakov monopole [18,19] or the self-dual $SU(2)$ Belavin-Polyakov-Schwartz-Tyupkin instanton [20]. The $SU(2)$ CSG soliton plays a similar building block role in the HSG theories.

In their rest frame, HSG solitons provide periodic time-dependent solutions describing the rotation in the internal $U(1)^{\times r_g}$ space and, in Section 6, we apply the Bohr-Sommerfeld

quantization rule to obtain the semi-classical spectrum. After quantization, the classical coupling constant of the theory becomes the level k of the underlying G-parafermionic theory, and the vector charge $\boldsymbol{Q}$ is restricted to take values in the root lattice of g, $\boldsymbol{\Lambda}_R$, modulo k times the co-root lattice, $\boldsymbol{\Lambda}_R^*$, which is precisely the discrete symmetry group of the Gepner theory of G-parafermions at level-k [12]. For each root, there is a finite tower of massive particles whose number depends on the length of the root. Unlike the SG kinks, CSG and HSG solitons do not have topological quantum numbers and, as a consequence, the fundamental particle associated with a root $\boldsymbol{\alpha}$ can be identified with the $\boldsymbol{Q} = \boldsymbol{\alpha}$ soliton itself. Therefore, the full spectrum of the HSG theories is described by solitons.

One of the most novel features of solitons captured by the HSG theories is the existence of unstable particles. In fact, when $r_g \geq 2$, only the soliton particles associated with the simple roots are stable, while the other are either unstable (resonances) or bound-states at threshold. An identical phenomenon occurs for monopoles and dyons in $N = 2$ and $N = 4$ supersymmetric gauge theories [19], which provides extra motivation for the overall aim of finding a factorizable S-matrix capable of describing the quantum scattering of the solitons constructed in this paper.

Our conclusions are presented in Section 7 where we also comment on the relation between solitons and parafermions. We have collected our conventions about Kac-Moody algebras and vertex operators in an appendix.

2. The Homogeneous sine-Gordon theories

The construction of the HSG theory associated with a given compact simple Lie group G starts with the choice of two constant elements $\Lambda_\pm$ in g, the Lie algebra of G. The condition that the resulting theory has a mass-gap requires $\Lambda_\pm$ to be semisimple and regular, which means that their centralizer in g is a Cartan subalgebra. Otherwise, the choice of $\Lambda_\pm$ is completely free and, hence, they can be considered as continuous vector couplings of the theory. To comply with the notation of [5], where more details can be found, we will refer to the centralizer of Λ_+ in g and to the corresponding abelian group as $g_0^0 = u(1)^{+r_g}$ and $G_0^0 = U(1)^{\times r_g}$, respectively, where r_g is the rank of G. The HSG theory is specified by the action

$$S[h, A_\pm] \;=\; \frac{1}{\beta^2}\Big\{ S_{\rm WZW}[h, A_\pm] \;-\; \int d^2x\, V(h) \Big\}\,. \tag{2.1}$$

Here, h is a bosonic field that takes values in G, $A_\pm$ are (non-dymanical) abelian gauge connections taking values in g_0^0, and $S_{\rm WZW}[h, A_\pm]$ is the gauged Wess-Zumino-Witten

action describing the $G/G_0^0 = G/U(1)^{\times r_g}$ coset model,

$$S_{\rm WZW}[h, A_\pm] \;=\; S_{\rm WZW}[h] \;+\; \frac{1}{\pi} \int d^2x \, \Big(-\langle A_+ \,,\, \partial_- h \, h^\dagger \rangle \\ + \; \langle \tau(A_-) \,,\, h^\dagger \, \partial_+ h \rangle \;+\; \langle h^\dagger \, A_+ \, h \,,\, \tau(A_-) \rangle \;-\; \langle A_+ \,,\, A_- \rangle \Big) \,, \tag{2.2}$$

where $x_\pm \;=\; t \pm x$ are light-cone variables in the $1+1$ Minkowski space. The potential $V(h)$ is

$$V(h) \;=\; -\,\frac{m^2}{\pi}\, \langle \Lambda_+ \,,\, h^\dagger \, \Lambda_- \, h \rangle \,, \tag{2.3}$$

and we have denoted the Killing form on g by $\langle \; , \; \rangle$. The normalization is chosen such that long roots have square length 2 and, hence, $S_{\rm WZW}[h]$ is uniquely defined modulo $2\pi\mathbb{Z}$ [21,22]. Finally, $\hbar m$ is a constant with dimensions of mass, and β is a coupling constant that has to be quantized if the quantum theory is to be well defined; namely, $\hbar\beta^2 = 1/k$, where k is a positive integer [22]. The Planck constant is explicitly shown to exhibit that, just as in the sine-Gordon theory, the semi-classical limit is the same as the weak-coupling limit, and that both are recovered when $k \to \infty$.

The action (2.1) is invariant with respect to the abelian gauge transformations

$$h(x,t) \mapsto {\rm e}^{\,\alpha}\, h \, {\rm e}^{\,-\,\tau(\alpha)} \,, \qquad A_\pm \mapsto A_\pm \;-\; \partial_\pm \alpha \,, \tag{2.4}$$

where $\alpha \;=\; \alpha(x,t)$ takes values in $g_0^0 \;=\; u(1)^{+r_g}$. The precise form of the group of gauge transformations is specified by τ, which is an automorphism of g_0^0 that is compatible with the Killing form of g. In our case, since g_0^0 is abelian and the restriction of $\langle \; , \; \rangle$ to g_0^0 is euclidean, τ is an element of the group $O(r_g)$. However, in order to ensure that the resulting theory has a mass-gap, τ has to be chosen as follows [5]. Let us consider the $x_\pm$-independent field configuration h_0 corresponding to the vacuum of the theory or, equivalently, to the absolute minimum of the potential. The vacuum configuration h_0 satisfies $[\Lambda_+, h_0^\dagger \, \Lambda_- \, h_0] = 0$ and, since $\Lambda_\pm$ are regular, it induces, by conjugation, an inner automorphism of G that fixes the subalgebra g_0^0, *i.e.*, $h_0^\dagger \, g_0^0 \, h_0 = g_0^0$. Then, the condition that the theory has a mass-gap requires that the automorphism τ is chosen such that

$$\tau(u) \neq h_0^\dagger \, u \, h_0 \tag{2.5}$$

for any u in g_0^0 [5]. Otherwise, the choice of τ is completely free and it provides supplementary continuous coupling constants, besides $\Lambda_\pm$.

The resulting HSG theory exhibits an abelian $U(1)^{\times r_g}$ global symmetry with respect to the group of transformations [1]

$$h(x,t) \mapsto {\rm e}^{\,\alpha}\, h \, {\rm e}^{\,-\,(h_0^\dagger \, \alpha \, h_0)} \,, \tag{2.6}$$

[1] Condition (2.5) allows one to split any global transformation $h \mapsto {\rm e}^{\,\alpha}\, h \, {\rm e}^{\,\beta}$ as the composition of a global gauge transformation (2.4) and a global symmetry transformation in a unique way.

where α is a constant, $x_\pm$-independent, element of g_0^0; these transformations fix the vacuum configuration h_0. In order to give the expression of the corresponding Noether current, let us briefly describe the gauge-fixing procedure. It can be viewed as the choice of some 'canonical' form $h^{\rm can}$ (gauge slice) such that any h can be taken to that form through a non-singular h-dependent gauge transformation,

$$h \mapsto h^{\rm can} \;=\; {\rm e}^{\,\phi^{\rm can}[h]}\, h\, {\rm e}^{\,-\tau(\phi^{\rm can}[h])}\,. \tag{2.7}$$

Then, the gauge invariant conserved Noether current is [5]

$$J^\mu \;=\; \epsilon^{\mu\,\nu}\,\left(A_\nu \,+\, \partial_\nu \phi^{\rm can}\right)\,. \tag{2.8}$$

In the following, we will assume that the vacuum configuration corresponds to $h_0 = I$ (up to gauge transformations), which can always be achieved through a change of variables $h \mapsto h_0 h$. This implies that $\tau \neq I$ and, hence, that the group of gauge transformations will not be of vector type while global transformations (2.6) will always be of vector type.

Concerning gauge-fixing, we will use the 'Leznov-Saveliev prescription' (LS) [5]. It consists in choosing $A_\pm = 0$, which specifies the form of $h^{\rm can}$ only up to global gauge transformations. This choice is viable due to the on-shell flatness of the gauge field considered on two-dimensional Minkowski space. In the LS gauge, the equations-of-motion reduce to

$$\partial_-\left(h^\dagger\partial_+ h\right) \;=\; -\,m^2\left[\Lambda_+\,,\, h^\dagger\Lambda_- h\right]\,, \tag{2.9}$$

$$P\big(h^\dagger\partial_+ h\big) \;=\; P\big(\partial_- h\, h^\dagger\big) \;=\; 0\,, \tag{2.10}$$

where P is the projector onto the subalgebra g_0^0. Eq. (2.9) is a non-abelian affine Toda equation [23,24,25], and (2.10) are the constraints that specify the form of the gauge slice in this case. These constraints cannot be solved locally [14]. However, using the method of Leznov and Saveliev, the explicit general solution of the non-abelian Toda equation along with the constraints (2.10) can be obtained in a systematic way by means of the representation theory of affine Kac-Moody algebras [23]. In particular, the multi-soliton solutions are obtained, in this gauge, by means of the so-called 'solitonic specialization' [15,16] or, equivalently, through the method of dressing transformations [17].

It is worth noticing that HSG theories do not exhibit parity invariance for generic values of the coupling constants $\Lambda_\pm$ and τ. To be specific, parity invariance requires that

$$\Lambda_+ \;=\; \mu\, h_0^\dagger \Lambda_- h_0 \quad \text{and} \quad \tau(u) \;=\; -\,h_0^\dagger\, u\, h_0\,, \tag{2.11}$$

for any u in g_0^0, and some real number μ [5]. Then the parity transformation is $x \mapsto -x$, $h \mapsto h_0 h^\dagger h_0$, and $A_\pm \mapsto -A_\pm$.

3. Multi-Soliton solutions of HSG theories

In this section, we work out the solitonic specialization of the Leznov-Saveliev solution corresponding to the equations (2.9) and (2.10) following the approach and conventions of [17], which is based on the method of dressing transformations. First of all, we have to exhibit the relationship of these equations with affine Kac-Moody algebras.

Let $g_{\mathbf{c}}$ be the complexification of the compact Lie algebra g, and consider the central extension of its loop algebra, $\widehat{g}$ (see the appendix). Following the approach of [17], the eqs. (2.9) and (2.10) are generalized non-abelian Toda equations associated with the homogeneous gradation $\mathbf{s}_{\rm h} = (1, 0, \ldots, 0)$ of the untwisted affine Kac-Moody algebra $g_{\mathbf{c}}^{(1)} = \widehat{g} \oplus \mathbb{C} d$ and with the vacuum solution specified by $l = 1$, $t_{\pm 1} = \pm x_{\pm 1}$,

$$\mathcal{A}_{1}^{(\rm vac)} \; = \; m\, \Lambda_{+}^{(1)} \, , \quad \text{and} \quad \mathcal{A}_{-1}^{(\rm vac)} \; = \; m\, \Lambda_{-}^{(-1)} \; + \; m^2 \, \langle \Lambda_{+} \, , \, \Lambda_{-} \rangle \, x_{+}\, c \tag{3.1}$$

(see Section IV.2 of [17], and recall that $[\Lambda_{+}^{(1)}, \Lambda_{-}^{(-1)}] = \langle \Lambda_{+}, \Lambda_{-} \rangle \, c$). Consider a field $B = B(x_+, x_-)$ taking values in the finite Lie group formed by exponentiating the subalgebra $\widehat{g}_{\,0}(\mathbf{s}_{\rm h})$. Since $\widehat{g}_{\,0}(\mathbf{s}_{\rm h}) = g_{\mathbf{c}} \oplus \mathbb{C}\, c$, the field B can be split as $B = h \, \exp(\nu\, c)$, where $\nu = \nu(x_+, x_-)$ is the component along the central element c and $h = h(x_+, x_-)$ takes values in $G_{\mathbf{c}}$, the complex Lie group formed by exponentiating $g_{\mathbf{c}}$. Then, the approach of [17] provides a class of solutions for the equation

$$\partial_{-} \left(B^{-1} \partial_{+} B\right) \; = \; -\, m^2 \Big(\big[\Lambda_{+}^{(1)} \, , \, B^{-1} \Lambda_{-}^{(-1)} B\big] \; - \; \langle \Lambda_{+} \, , \, \Lambda_{-} \rangle \, c \Big) \, , \tag{3.2}$$

subject to the constraints

$$B^{-1} \partial_{+} B \; \in \; {\rm Im} \Big({\rm ad}\, \Lambda_{+}^{(1)} \Big) \; , \quad \text{and} \quad \partial_{-} B \, B^{-1} \; \in \; {\rm Im} \Big({\rm ad}\, \Lambda_{-}^{(-1)} \Big) \; . \tag{3.3}$$

Taking into account $B = h \, \exp(\nu\, c)$, eq. (3.2) can be split as

$$\partial_{-} \left(h^{-1} \partial_{+} h\right) \; = \; -\, m^2 \big[\Lambda_{+} \, , \, h^{-1} \Lambda_{-} h\big] \, , \tag{3.4}$$

$$\partial_{-} \partial_{+} \nu \; = \; -\, m^2 \, \langle \Lambda_{+} \, , \, h^{-1} \Lambda_{-} h \; - \; \Lambda_{-} \rangle \, , \tag{3.5}$$

while the constraints (3.3) become

$$P_{\mathbf{c}}(h^{-1} \partial_{+} h) \; = P_{\mathbf{c}}(\partial_{-} h \, h^{-1}) \; = \; 0 \, , \tag{3.6}$$

where $P_{\mathbf{c}}$ is the projector onto the centralizer of $\Lambda_{\pm}$ in $g_{\mathbf{c}}$.

The resulting solutions are expressed by means of the integrable highest-weight representations of $g_{\mathbf{c}}^{(1)}$ (see the appendix for the notation). Since these equations are related to the homogeneous gradation $\mathbf{s}_{\rm h}$, one needs the fundamental representation $L(0)$, whose highest-weight vector is $|v_0\rangle$, to identify the value of ν, and another highest-weight representation $L(\tilde{\mathbf{s}})$ such that the subset of vectors which are annihilated by all the elements in $\widehat{g}_{>0}(\mathbf{s}_{\rm h})$ form a faithful representation of $G_{\mathbf{c}}$. Then, these vectors can be labelled by means of the weights of the representation, $|\boldsymbol{\mu}_0\rangle, |\boldsymbol{\mu}_0'\rangle$, and the solutions are given by [17]

$$\langle\boldsymbol{\mu}_0'|h^{-1}|\boldsymbol{\mu}_0\rangle \;=\; \frac{\langle\boldsymbol{\mu}_0'|{\rm e}^{\,m\,x_+\Lambda_+^{(1)}}\;{\rm e}^{\,-\,m\,x_-\Lambda_-^{(-1)}}\;b\,{\rm e}^{\,m\,x_-\Lambda_-^{(-1)}}\;{\rm e}^{\,-\,m\,x_+\Lambda_+^{(1)}}|\boldsymbol{\mu}_0\rangle}{\langle v_0|{\rm e}^{\,m\,x_+\Lambda_+^{(1)}}\;{\rm e}^{\,-\,m\,x_-\Lambda_-^{(-1)}}\;b\,{\rm e}^{\,m\,x_-\Lambda_-^{(-1)}}\;{\rm e}^{\,-\,m\,x_+\Lambda_+^{(1)}}|v_0\rangle}\,, \tag{3.7}$$

$${\rm e}^{-\nu} \;=\; \langle v_0|{\rm e}^{\,m\,x_+\Lambda_+^{(1)}}\;{\rm e}^{\,-\,m\,x_-\Lambda_-^{(-1)}}\;b\,{\rm e}^{\,m\,x_-\Lambda_-^{(-1)}}\;{\rm e}^{\,-\,m\,x_+\Lambda_+^{(1)}}|v_0\rangle \;\equiv\; \tau_0\,, \tag{3.8}$$

for any constant element b in the Kac-Moody group associated with $g_{\mathbf{c}}^{(1)}$.

The class of solutions summarized by eqs. (3.8) and (3.7) are conjectured to include the multi-soliton solutions [15,16,17,24]. They should correspond to group elements which are products of exponentials of common eigenvectors of $\Lambda_+^{(1)}$ and $\Lambda_-^{(-1)}$:

$$b \;=\; {\rm e}^{F_1}\,{\rm e}^{F_2}\;\cdots\,, \qquad [\Lambda_\pm^{(\pm1)}\,,\,F_k] = \omega_\pm^{(k)}\,F_k\,. \tag{3.9}$$

In our case, and using the Chevalley basis introduced in the appendix, these eigenvectors are associated with the roots of g. Namely, for each root $\boldsymbol{\alpha}$ and complex number z,

$$F_{\boldsymbol{\alpha}}(z) \;=\; \sum_{n\in\mathbb{Z}} z^{-n}\,E_{\boldsymbol{\alpha}}^{(n)} \tag{3.10}$$

is an eigenvector of $\Lambda_\pm^{(\pm1)} = \pm\, i\boldsymbol{\lambda}_\pm\cdot\boldsymbol{h}^{(\pm1)}$ whose eigenvalues are

$$\omega_\pm^{\boldsymbol{\alpha}}(z) \;=\; \pm\, i\, z^{\,\pm1}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_\pm)\,. \tag{3.11}$$

In these equations, $\boldsymbol{\lambda}_\pm$ are the components of $\Lambda_\pm$ in that basis, and they have to be chosen in the same Weyl chamber of the Cartan subalgebra to ensure that $h_0 = I$ actually corresponds to the minimum of $V(h)$. Recall that the fundamental particles are associated with the roots of g, and the mass of the particle associated with $\boldsymbol{\alpha}$ is [5]

$$m_{\boldsymbol{\alpha}} \;=\; 2m\,\sqrt{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_+)\,(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_-)}\,. \tag{3.12}$$

Eqs. (3.4) and (3.6) are identical to the equations-of-motion of the HSG theories in the LS gauge; namely, eqs. (2.9) and (2.10), respectively. However, in the former h takes values in the complex Lie group $G_{\mathbf{c}}$, while in the original problem it has to take values in

the compact Lie group G. Therefore, it is necessary to restrict the choice of the arbitrary group element b to ensure that the resulting solutions for h take values in G. Recall that the elements of $g \subset g_{\mathbf{c}}$ are characterized by means of the anti-linear involutive automorphism θ defined by eq. (A.2). In a similar fashion, the elements of the central extension of $g_{\mathbf{c}}$ which are compatible with the condition that $\widehat{g}_{\,0}(\mathbf{s}_{\rm h})\,\big|_{c=0}$ is compact can be singled out through the following extension of θ to $\widehat{g}$: [2]

$$\hat{\theta}\Big(u^{(m)}\Big) \;=\; \Big(\theta(u)\Big)^{(m)}\,, \quad \hat{\theta}(c) \;=\; c\,. \tag{3.13}$$

This automorphism of $\widehat{g}$ has the characteristic property of preserving the homogeneous gradation, $\hat{\theta}\big(\,\widehat{g}_{\,n}(\mathbf{s}_{\rm h})\big) \;=\; \widehat{g}_{\,n}(\mathbf{s}_{\rm h})$, which is needed because, in this case, homogeneous grades characterize transformation properties with respect to the two-dimensional Lorentz group. Then, the condition that products of the form (3.9) are fixed by $\hat{\theta}$ requires that, instead of exponentials of eigenvectors, one has to consider exponentials of combinations of the form

$$a\, F_{\boldsymbol{\alpha}}(z) \;-\; a^{*}\, F_{\boldsymbol{\alpha}}^{\dagger}(z)\,, \tag{3.14}$$

for each root $\boldsymbol{\alpha}$ and complex numbers a and z. Since these elements are not common eigenvectors of $\Lambda_{\pm}^{(\pm1)}$, this prescription can be somehow considered as a generalization of the original solitonic specialization designed to ensure that the resulting solutions take values in a compact group. Therefore, we conjecture that all the multi-soliton solutions of eq. (2.9), constrained by (2.10), can be obtained from eq. (3.7) by considering group elements b which are products of exponentials of elements of the form given by eq. (3.14).

In the LS gauge, the determination of the mass and charge carried by the solitons is greatly simplified, and their relation with the boundary conditions becomes explicit. Since the HSG theories exhibit a unique vacuum, the boundary condition satisfied by soliton solutions at $x \to \pm\infty$ is that their field configurations become h_0, up to gauge transformations. However, using the LS gauge-fixing prescription $A_{\pm} = 0$, the only remnant gauge freedom corresponds to global transformations and, hence, solitons satisfy

$$\begin{aligned} \lim_{x\to\pm\infty} h(x,t) \;&=\; {\rm e}^{\,q_{\pm}}\, h_0\, {\rm e}^{\,-\tau(q_{\pm})} \\ &=\; {\rm e}^{\,q_{\pm} \,-\, h_0 \tau(q_{\pm}) h_0^{\dagger}}\, h_0 \;=\; {\rm e}^{\,\xi_{\pm}}\, h_0\,, \end{aligned} \tag{3.15}$$

where $q_{\pm}$ are constant elements in g_0^0.

[2] Notice that $\hat{\theta}$ is not the 'Chevalley involution' of $g_{\mathbf{c}}^{(1)}$, which is determined by $\omega(E_{\boldsymbol{\alpha}_i}^{(0)}) = -E_{-\boldsymbol{\alpha}_i}^{(0)}$, for $i = 1,\ldots,r_g$, and $\omega(E_{\boldsymbol{\alpha}_0}^{(+1)}) = -E_{-\boldsymbol{\alpha}_0}^{(-1)}$ and, hence, $\omega(\,\widehat{g}_{\,n}(\mathbf{s}_{\rm h})) = \widehat{g}_{\,-n}(\mathbf{s}_{\rm h})$.

Let us first consider the calculation of the $U(1)^{\times r_g}$ charge. In the LS gauge, the conserved Noether current (2.8) simplifies to $J^\mu = \epsilon^{\mu\,\nu}\partial_\nu \phi^{\rm can}$ and, hence, the corresponding conserved charge is [3]

$$i\,\boldsymbol{Q}\cdot\boldsymbol{h} \;=\; \int_{-\infty}^{+\infty} dx\; J^0 \;=\; \phi^{\rm can}[h(-\infty,t)] \;-\; \phi^{\rm can}[h(+\infty,t)]\,. \tag{3.16}$$

Then, taking into account the definition of $\phi^{\rm can} = \phi^{\rm can}[h]$, eq. (2.7), and the boundary condition (3.15), the value of the $U(1)^{\times r_g}$ vector charge is $q_- - q_+$. However, the condition (2.5) allows one to define the conserved charge as

$$i\,\boldsymbol{Q}\cdot\boldsymbol{h} \;\equiv\; \frac{1}{2\pi\beta^2}\,(\xi_+ \;-\; \xi_-)\,, \tag{3.17}$$

which is easier to derive directly from the asymptotic behaviour of soliton solutions in practice, and where the normalization has been chosen to simplify the results of Section 6.

Next, let us address the calculation of the energy-momentum tensor, which provides the mass of the solitons. Since the energy-momentum tensor is gauge-invariant [5], it can be calculated directly in the LS gauge where the equations-of-motion of the HSG theories become non-abelian affine Toda equations. Then, the calculation is largely simplified by considering the results of [24], where it is shown that the energy-momentum tensor of the theory splits in two pieces

$$\begin{aligned} T_{\mu\,\nu} \;&=\; \Theta_{\mu\,\nu} \;-\; \frac{1}{2\pi\beta^2}\,\bigl(\partial_\mu\partial_\nu \;-\; g_{\mu\,\nu}\,\partial_\rho\partial^\rho\bigr)\,\widehat{\nu}\,, \\ \text{where}\quad \widehat{\nu} \;&=\; \nu \;-\; m^2\,\langle\Lambda_+\,,\,\Lambda_-\rangle\, x_+x_-\,. \end{aligned} \tag{3.18}$$

This is the generalization to the non-abelian case of a similar expression originally obtained in the context of abelian affine Toda theories [15,26]. In (3.18), $\Theta_{\mu\,\nu}$ is the traceless energy-momentum tensor of a related conformal affine Toda theory associated with the Kac-Moody algebra $g_{\mathbf{c}}^{(1)}$, which vanishes for multi-soliton solutions [24,15,26]. This means that the energy and momentum carried by solitons are given by the second term in (3.18) and, therefore, they arise through the boundary values of $\nu = \nu(x,t)$, the component of the field B along the central element of $g_{\mathbf{c}}^{(1)}$ (see eq. (3.2)). Namely,

$$P \;=\; \int_{-\infty}^{+\infty} dx\, T_{01} \;=\; \frac{1}{2\pi\beta^2}\,\bigl(\partial_t\nu(-\infty,t) \;-\; \partial_t\nu(+\infty,t)\bigr)\,, \tag{3.19}$$

$$E \;=\; \int_{-\infty}^{+\infty} dx\,\Bigl(T_{00} \;-\; \frac{1}{\beta^2}V(h_0)\Bigr) \;=\; \frac{1}{2\pi\beta^2}\,\bigl(\partial_x\nu(-\infty,t) \;-\; \partial_x\nu(+\infty,t)\bigr)\,, \tag{3.20}$$

where the value of $V(h_0)/\beta^2$ has been subtracted to ensure that the vacuum configuration has $E = P = 0$.

[3] Our convention for the antisymmetric tensor is $\epsilon_{01} = \epsilon^{10} = +1$.

4. Soliton solutions of simply-laced HSG theories

In this section we analyze the soliton solutions of the HSG theories associated with simply-laced Lie algebras. In this case, the soliton solutions can be expressed by means of the level-one fundamental representations of $g_{\mathbf{c}}^{(1)}$ realized through the homogeneous vertex operator construction [27,28,29] (see the appendix). Therefore, in eq. (3.10), $F_{\boldsymbol{\alpha}}(z) = V_{\boldsymbol{\alpha}}(z)$, and the multi-soliton solutions are obtained from eq. (3.7) by considering group elements b which are products of elements of the form [4]

$$
\begin{aligned}
b_{\boldsymbol{\alpha}}(a,z) \;&=\; \mathrm{e}^{\,a\, V_{\boldsymbol{\alpha}}(z) \,+\, a^*\, V_{-\boldsymbol{\alpha}}(z^*)} \;=\; \big(1 \,+\, a\, V_{\boldsymbol{\alpha}}(z)\big)\,\big(1 \,+\, a^*\, V_{-\boldsymbol{\alpha}}(z^*)\big) \\
&=\; 1 \,+\, a\, V_{\boldsymbol{\alpha}}(z) \,+\, a^*\, V_{-\boldsymbol{\alpha}}(z^*) \,+\, |a|^2\, V_{\boldsymbol{\alpha}}(z)\, V_{-\boldsymbol{\alpha}}(z^*)\,,
\end{aligned}
\tag{4.1}
$$

where we have used the widely known nilpotency of $V_{\boldsymbol{\alpha}}(z)$, eq. (A.17). Notice that, since $b_{-\boldsymbol{\alpha}}(a,z) = b_{\boldsymbol{\alpha}}(a^*,z^*)$ and

$$
a \;=\; |a|\, \mathrm{e}^{\,i\phi} \quad \text{and} \quad z \;=\; \mathrm{e}^{\,\rho \,+\, i\varphi}
\tag{4.2}
$$

are generic complex numbers, we can assume that $\boldsymbol{\alpha}$ is always a positive root and that $(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_{\pm}) > 0$ (recall that $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$ have to be chosen in the same Weyl chamber). In eq. (3.7), the only effect of the conjugation of $b_{\boldsymbol{\alpha}}(a,z)$ with $\mathrm{e}^{\,m\,x_+ \Lambda_+^{(1)}}\, \mathrm{e}^{\,-\,m\,x_- \Lambda_-^{(-1)}}$ is the change

$$
a \;\mapsto\; d_{\boldsymbol{\alpha}}(z) \;=\; a\, \mathrm{e}^{\,\Gamma_{\boldsymbol{\alpha}}(z)}
\tag{4.3}
$$

where

$$
\begin{aligned}
\Gamma_{\boldsymbol{\alpha}}(z) \;&=\; m\, x_+\, \omega_+^{\boldsymbol{\alpha}}(z) \;-\; m\, x_-\, \omega_-^{\boldsymbol{\alpha}}(z) \\
&=\; -m_{\boldsymbol{\alpha}}\, \sin(\varphi)\, \frac{x \,+\, v_{\boldsymbol{\alpha}}\, t}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}} \;+\; i\, m_{\boldsymbol{\alpha}}\, \cos(\varphi)\, \frac{t \,+\, v_{\boldsymbol{\alpha}}\, x}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}}\;,
\end{aligned}
\tag{4.4}
$$

where $m_{\boldsymbol{\alpha}}$ is the mass of the fundamental particle associated with $\boldsymbol{\alpha}$ and

$$
v_{\boldsymbol{\alpha}} \;=\; \tanh\Big[\rho \,+\, \frac{1}{2}\, \ln \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_+)}{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_-)}\Big]\,.
\tag{4.5}
$$

There is a one-soliton solution associated with each (positive) root $\boldsymbol{\alpha}$ of g that is recovered with $b = b_{\boldsymbol{\alpha}}(a,z)$. Its explicit form can be easily obtained from eqs. (3.7),

[4] Since we follow the conventions of [29] for vertex operators, notice that $E_{\pm\boldsymbol{\alpha}}^{\dagger} = -E_{\mp\boldsymbol{\alpha}}$ and, hence, $F_{\boldsymbol{\alpha}}^{\dagger}(z) = -V_{-\boldsymbol{\alpha}}(z^*)$ (see the appendix).

(A.10), and (A.16):

$$
\begin{aligned}
\langle \boldsymbol{\mu}_0' | h^\dagger | \boldsymbol{\mu}_0 \rangle \;=\; \frac{1}{\tau_0} \Bigg(& \Big[1 \,+\, \frac{|z|^2 \, |a|^2}{|z - z^*|^2} \, \mathrm{e}^{\,\Gamma_{\boldsymbol{\alpha}}(z) \,+\, \Gamma_{\boldsymbol{\alpha}}^*(z)} \, \Big(\frac{z}{z^*}\Big)^{\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0} \Big] \, \delta_{\boldsymbol{\mu}_0, \boldsymbol{\mu}_0'} \\
& +\, \epsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}_0 - v_{\tilde{\mathbf{s}}}) \, z^{1 \,+\, \boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0} \, a \, \mathrm{e}^{\,\Gamma_{\boldsymbol{\alpha}}(z)} \, \delta_{\boldsymbol{\mu}_0 + \boldsymbol{\alpha}, \boldsymbol{\mu}_0'} \\
& +\, \epsilon(-\boldsymbol{\alpha}, \boldsymbol{\mu}_0 - v_{\tilde{\mathbf{s}}}) \, (z^*)^{1 \,-\, \boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0} \, a^* \, \mathrm{e}^{\,\Gamma_{\boldsymbol{\alpha}}^*(z)} \, \delta_{\boldsymbol{\mu}_0, \boldsymbol{\mu}_0' + \boldsymbol{\alpha}} \Bigg) \,,
\end{aligned}
\tag{4.6}
$$

where $v_{\tilde{\mathbf{s}}}$ is the highest-weight of the representation $L(\tilde{\mathbf{s}})$ involved in eq. (3.7), and

$$
\begin{aligned}
\tau_0 \;=\; \mathrm{e}^{\,-\nu} \;&=\; 1 \,+\, \frac{|z|^2 \, |a|^2}{|z - z^*|^2} \, \mathrm{e}^{\,\Gamma_{\boldsymbol{\alpha}}(z) \,+\, \Gamma_{\boldsymbol{\alpha}}^*(z)} \\
&=\; 1 \,+\, \frac{|a|^2}{4 \, \sin^2 \varphi} \, \exp\Big[-2 m_{\boldsymbol{\alpha}} \, \sin(\varphi) \, \frac{x \,+\, v_{\boldsymbol{\alpha}} \, t}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}} \Big] \,.
\end{aligned}
\tag{4.7}
$$

In level-one representations of $g_{\mathbf{c}}^{(1)}$ where g is simply-laced, all the weights $\boldsymbol{\mu}_0$ satisfy $\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0 = 0, \pm 1$ [28] [5] and, hence, that $E_{\boldsymbol{\alpha}}^{(0)} | \boldsymbol{\mu}_0 \rangle \neq 0$ if, and only if, $\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0 = -1$. Then, the only non-vanishing matrix elements of $h^\dagger$ are

$$
\begin{aligned}
\langle \boldsymbol{\mu}_0 + \boldsymbol{\alpha} | h^\dagger | \boldsymbol{\mu}_0 \rangle \;&=\; -\, \epsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}_0 - v_{\tilde{\mathbf{s}}}) \, u_{\boldsymbol{\alpha}}^*(x,t) \\
\langle \boldsymbol{\mu}_0 | h^\dagger | \boldsymbol{\mu}_0 + \boldsymbol{\alpha} \rangle \;&=\; \epsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}_0 - v_{\tilde{\mathbf{s}}}) \, u_{\boldsymbol{\alpha}}(x,t) \,, \quad \text{if} \quad \boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0 = -1 \,,
\end{aligned}
\tag{4.8}
$$

and

$$
\langle \boldsymbol{\mu}_0 | h^\dagger | \boldsymbol{\mu}_0 \rangle \;=\; \sqrt{1 \,-\, |(\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0) \, u_{\boldsymbol{\alpha}}(x,t)|^2} \;\, \mathrm{e}^{\,-\, i \, (\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0) \, \eta_{\boldsymbol{\alpha}}(x,t)} \,, \qquad \text{for all} \quad \boldsymbol{\mu}_0 \,, \tag{4.9}
$$

where

$$
u_{\boldsymbol{\alpha}}(x,t) \;=\; -\, |\sin\varphi| \, \frac{\mathrm{e}^{\,-\, i\,\phi_0} \, \exp\Big(-\, i \, m_{\boldsymbol{\alpha}} \cos\varphi \, \frac{t \,+\, v_{\boldsymbol{\alpha}} (x \,-\, x_0)}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}} \Big)}{\cosh\Big(m_{\boldsymbol{\alpha}} \, \sin\varphi \, \frac{x \,-\, x_0 \,+\, v_{\boldsymbol{\alpha}} \, t}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}} \Big)} \,, \quad \text{and} \tag{4.10}
$$

$$
\eta_{\boldsymbol{\alpha}}(x,t) \;=\; -\, \varphi \,+\, \arctan\Big[\tan\varphi \, \tanh\Big(m_{\boldsymbol{\alpha}} \, \sin\varphi \, \frac{x \,-\, x_0 \,+\, v_{\boldsymbol{\alpha}} \, t}{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}} \Big) \Big] \,. \tag{4.11}
$$

In other words, h takes values in the representation of the regular $SU(2)$ subgroup of G generated by $E_{\pm\boldsymbol{\alpha}}^{(0)}$ provided by the vertex operator construction, an observation whose importance will become clear in the next Section. This representation is formed by the set of vectors $|\boldsymbol{\mu}_0\rangle$ and $|\boldsymbol{\mu}_0 + \boldsymbol{\alpha}\rangle$ with $\boldsymbol{\alpha}\cdot\boldsymbol{\mu}_0 = -1$, which is a direct sum of spin-1/2 irreducible factors. In the last equations we have introduced

$$
x_0 \;=\; \frac{\sqrt{1 \,-\, v_{\boldsymbol{\alpha}}^2}}{m_{\boldsymbol{\alpha}} \, \sin\varphi} \, \ln\Big| \frac{a}{2\, \sin\varphi} \Big| \qquad \text{and} \qquad \phi_0 \;=\; \phi \,+\, \frac{v_{\boldsymbol{\alpha}}}{\tan\varphi} \, \ln\Big| \frac{a}{2\, \sin\varphi} \Big| \,, \tag{4.12}
$$

[5] This property easily follows from $\| E_{\pm\boldsymbol{\alpha}}^{(-1)} | \boldsymbol{\mu}_0 \rangle \|^2 = \langle \boldsymbol{\mu}_0 | E_{\pm\boldsymbol{\alpha}}^{\dagger (1)} E_{\pm\boldsymbol{\alpha}}^{(-1)} | \boldsymbol{\mu}_0 \rangle \geq 0$ and $E_{\pm\boldsymbol{\alpha}}^{(1)} | \boldsymbol{\mu}_0 \rangle = 0$.

which correspond to the centre of mass of the soliton and its orientation in the internal $U(1)^{\times r_g}$ space.

Using eqs. (3.19) and (3.20), the energy and momentum carried by this soliton can be easily calculated from the asymptotic behaviour of τ_0, eq. (4.7):

$$P \;=\; E\, v_{\boldsymbol{\alpha}} \;=\; \frac{1}{\pi\beta^2}\, \frac{m_{\boldsymbol{\alpha}}\, |\sin\varphi|}{\sqrt{1\,-\,v_{\boldsymbol{\alpha}}^2}}\, v_{\boldsymbol{\alpha}}\,. \tag{4.13}$$

Therefore, the configuration given by eq. (4.6) actually corresponds to a relativistic soliton with mass

$$M_{\boldsymbol{\alpha}}(\varphi) \;=\; \frac{1}{\pi\beta^2}\, m_{\boldsymbol{\alpha}}\, |\sin\varphi| \tag{4.14}$$

moving with velocity $v_{\boldsymbol{\alpha}}$; notice that $|v_{\boldsymbol{\alpha}}|$ is always < 1. In their rest frame, $v_{\boldsymbol{\alpha}} = 0$, these field configurations are not static but periodic time-dependent solutions that rotate in the internal $U(1)^{\times r_g}$ space with angular velocity

$$\omega_{\boldsymbol{\alpha}}(\varphi) \;=\; m_{\boldsymbol{\alpha}}\, \cos\varphi\,. \tag{4.15}$$

Taking eq. (3.17) into account, the charge carried by the soliton can be obtained from the asymptotic behaviour of its field configuration:

$$h^{\dagger}(\pm\infty, t) \;=\; \begin{cases} I & ,\ \text{if}\ \pm\sin\varphi > 0 \\ \mathrm{e}^{\,2\,i\,\varphi\,\boldsymbol{\alpha}\cdot\boldsymbol{h}} & ,\ \text{if}\ \pm\sin\varphi < 0\ . \end{cases} \tag{4.16}$$

Therefore, the conserved charged carried by the soliton is

$$\boldsymbol{Q}_{\boldsymbol{\alpha}}(\varphi) \;=\; \frac{1}{\pi\beta^2}\, \mathrm{sign}[\,\sin\varphi\,]\, \varphi\, \boldsymbol{\alpha} \quad \mathrm{mod} \quad \frac{1}{\beta^2}\, \boldsymbol{\Lambda}_R\,, \tag{4.17}$$

which is uniquely defined modulo $1/\beta^2$ times any element of $\boldsymbol{\Lambda}_R$, the root lattice of g; an ambiguity that does not modify the asymptotic behaviour given by (4.16) (recall that g is simply-laced).

We can summarize the physical meaning of the two complex parameters $a = |a|\, \mathrm{e}^{\,i\phi}$ and $z \;=\; \mathrm{e}^{\,\rho+i\varphi}$ that label the soliton solutions associated with a (positive) root $\boldsymbol{\alpha}$ as follows. Firstly, eq. (4.12) shows that the only role of $|a|$ and ϕ is to specify the position of the centre of mass of the soliton and its orientation in the internal space, respectively. Actually, the only effect of a global symmetry transformation $h \mapsto \mathrm{e}^{\,i\,\boldsymbol{\mu}\cdot\boldsymbol{h}}\, h\, \mathrm{e}^{-i\,\boldsymbol{\mu}\cdot\boldsymbol{h}}$ is the shift $\phi \mapsto \phi + \boldsymbol{\mu}\cdot\boldsymbol{\alpha}$, which is equivalent to a translation in the internal space. Secondly, and according to eq. (4.5), ρ is the rapidity of the soliton up to a constant, and φ specifies the angular velocity of the soliton motion in the internal $U(1)^{\times r_g}$ space together with its mass and charge.

Finally, let us investigate the range of values of φ leading to inequivalent soliton solutions. The transformation $\varphi \mapsto -\varphi$ is equivalent to $\eta_{\boldsymbol{\alpha}}(x,t) \mapsto \eta_{\boldsymbol{\alpha}}(x,t) - 2\varphi$ plus a change in x_0 and ϕ_0 that can be trivially absorbed in a. The change $\eta_{\boldsymbol{\alpha}}(x,t) \mapsto \eta_{\boldsymbol{\alpha}}(x,t) - 2\varphi$ can be induced by means of the axial-like global transformation $h \mapsto \mathrm{e}^{\,i\,\varphi\,\boldsymbol{\alpha}\cdot\boldsymbol{h}}\, h\, \mathrm{e}^{i\,\varphi\,\boldsymbol{\alpha}\cdot\boldsymbol{h}}$ that, taking eq. (2.5) into account, can always be split as the composition of a global gauge transformation and a global symmetry transformation. This implies that the field configurations corresponding to φ and $-\varphi$ actually correspond to the same soliton. Moreover, the vacuum configuration $h_0 = I$ is recovered with $\varphi = 0,\, \pi$. Taking into account all this and the fact that φ is an angular variable, we conclude that all the inequivalent soliton solutions are associated with the values of $\varphi \neq 0$ modulo π or, equivalent, with $\varphi \in (0,\pi)$.

5. Soliton solutions of arbitrary HSG theories

The soliton solutions of the HSG theories corresponding to non-simply laced algebras can be explicitly studied by means of appropriated 'foldings' of the vertex operator construction considered in the previous section. However, the properties of soliton solutions are better understood by noticing that eq. (4.10) provides just the one-soliton solutions of the complex sine-Gordon equation (CSG) [30,31,32]

$$\partial_+\partial_- u \;+\; u^* \,\frac{\partial_+ u\, \partial_- u^*}{1-|u|^2} \;+\; \frac{m_{\boldsymbol{\alpha}}^2}{4}\, u\left(1-|u|^2\right) \;=\; 0\,, \tag{5.1}$$

whose relation with the integrable deformation of the $SU(2)/U(1)$ coset conformal field theory by the first thermal operator was originally pointed out by Bakas [13] (see also [14]). Actually, the CSG theory is nothing else but the HSG associated with $SU(2)$, and we have already indicated that the soliton solutions constructed in the previous section are described by field configurations h taking values in a regular $SU(2)$ subgroup of G (see the comments below eq. (4.11)). Then, using the fundamental representation of $SU(2)$, the field can be parameterized as

$$h \;=\; \begin{pmatrix} \mathrm{e}^{\,i\,\eta}\,\sqrt{1-|u|^2} & -\,u^* \\ u & \mathrm{e}^{\,-i\,\eta}\,\sqrt{1-|u|^2} \end{pmatrix}\,, \tag{5.2}$$

and eq. (5.1) is the equation-of-motion of (2.1) with a particular gauge fixing [13,14,32].

In the original references, the CSG equation was associated with the Lagrangian [33,30,31]

$$\mathcal{L} \;=\; \frac{|\partial_\mu u|^2}{1-|u|^2} \;-\; m_{\boldsymbol{\alpha}}^2\, |u|^2\,, \tag{5.3}$$

whose relation with the action (2.1) is

$$S^{\mathrm{CSG}}[h, A_\pm] \;=\; \frac{1}{4\pi\beta^2}\int d^2x\,\mathcal{L} \tag{5.4}$$

Again, this relation involves a particular choice of the gauge fixing prescription such that the resulting form of the action is a local functional of u. Eq. (5.3) allows one to calculate the mass and charge of $SU(2)$ solitons directly by means of the Noether theorem:

$$\begin{aligned}
\frac{M^{\mathrm{CSG}}}{\sqrt{1\,-\,v_{\boldsymbol{\alpha}}^2}} \;&=\; \frac{1}{4\pi\beta^2}\int_{-\infty}^{+\infty} dx\,\Big(\frac{|\partial_t u|^2\,+\,|\partial_x u|^2}{1-|u|^2}\;+\;m_{\boldsymbol{\alpha}}^2\,|u|^2\Big)\,,\\
q \;&=\; \frac{i}{4\pi\beta^2}\int_{-\infty}^{+\infty} dx\,\frac{u^*\,\partial_t u\;-\;u\,\partial_t u^*}{1-|u|^2}\,.
\end{aligned} \tag{5.5}$$

Evaluated on the time-dependent soliton solutions (4.10), this yields eq. (4.14) for the mass, while the charge is given by

$$\begin{aligned}
q(\varphi) \;&=\; q(-\varphi) \;=\; (-1)^n\,q(\varphi+n\pi) \quad \text{and}\\
q(\varphi) \;&=\; \frac{1}{\pi\beta^2}\Big[\varphi\,+\,\frac{\pi}{2}\,\big(\mathrm{sign}[\frac{\pi}{2}-\varphi]\,-1\big)\Big] \quad \text{for} \quad \varphi\in(0,\pi)\,.
\end{aligned} \tag{5.6}$$

Although this might provide an unambiguous value for the charge carried by the soliton, we will keep the definition of the conserved charge given by eq. (4.17) and, hence,

$$\boldsymbol{Q}^{\mathrm{CSG}}(\varphi)\cdot\boldsymbol{h} \;=\; \big(q(\varphi)\mod\frac{1}{\beta^2}\big)\sigma_3\,, \tag{5.7}$$

where σ_1, σ_2, and σ_3 are the Pauli matrices. This definition of the charge emphasizes its interpretation as an angular variable, which is natural in the approach of the previous sections (see eq. (2.8)). In the following, we will also need the value of the action corresponding to the one-soliton solutions (4.10):

$$S^{\mathrm{CSG}}(\varphi) \;=\; \frac{2}{\beta^2}\Big[\big|\,\pi\beta^2\,q(\varphi)\,\big|\;-|\tan\varphi|\Big]\,. \tag{5.8}$$

Eq. (5.2) provides the general form of the field h in the fundamental representation of $SU(2)$ or, in the context of eqs. (3.7), (4.8), and (4.9), in the fundamental representation $L(\tilde{\mathbf{s}})=L(1)$ of $A_1^{(1)}$. Let us introduce a different but equivalent parameterization of h in the same representation

$$h \;=\; \mathrm{e}^{\,i\,\frac{\eta}{2}\,\sigma_3}\,\mathrm{e}^{\,i\,\psi[\cos\theta\,\sigma_1\,+\,\sin\theta\,\sigma_2]}\,\mathrm{e}^{\,i\,\frac{\eta}{2}\,\sigma_3} \;=\; \begin{pmatrix} \mathrm{e}^{\,i\,\eta}\,\cos\psi & i\sin\psi\,\mathrm{e}^{\,-i\,\theta}\\ i\sin\psi\,\mathrm{e}^{\,i\,\theta} & \mathrm{e}^{\,-i\,\eta}\,\cos\psi\end{pmatrix}\,. \tag{5.9}$$

It can be generalized to any representation of $SU(2)$ by means of

$$h \;=\; \mathrm{e}^{\,i\,\frac{\eta}{2}\,J_0}\,\mathrm{e}^{\,i\,\psi[\cos\theta\,(J_+ + J_-)\,-\,i\,\sin\theta\,(J_+ - J_-)]}\,\mathrm{e}^{\,i\,\frac{\eta}{2}\,J_0}\,, \tag{5.10}$$

where J_0 and $J_\pm$ are the Chevalley generators or $su(2)$, *i.e.*,

$$[J_0, J_\pm] = \pm 2 J_\pm \quad \text{and} \quad [J_+, J_-] = J_0 \,. \tag{5.11}$$

Considering eq. (5.10), the one-soliton solutions of the HSG theory associated with an arbitrary simple Lie group G can be obtained in the following way. Let $\boldsymbol{\alpha}$ be a positive root of g and construct the CSG one-soliton field configuration described, in the fundamental representation of $SU(2)$, by

$$h_{\boldsymbol{\alpha}}^{\rm CSG} \;=\; \begin{pmatrix} {\rm e}^{\,i\,\eta_{\boldsymbol{\alpha}}}\,\sqrt{1-|u_{\boldsymbol{\alpha}}|^2} & -\,u_{\boldsymbol{\alpha}}^* \\ u_{\boldsymbol{\alpha}} & {\rm e}^{\,-i\,\eta_{\boldsymbol{\alpha}}}\,\sqrt{1-|u_{\boldsymbol{\alpha}}|^2} \end{pmatrix} \,, \tag{5.12}$$

where $u_{\boldsymbol{\alpha}}$ and $\eta_{\boldsymbol{\alpha}}$ are given by eqs. (4.10) and (4.11), respectively. Next, let us consider the regular embedding of $SU(2)$ in G defined through

$$\iota_{\boldsymbol{\alpha}} \;:\; (J_+,\; J_-,\; J_0) \in su(2)_{\mathbf{c}} \longmapsto \Big(E_{\boldsymbol{\alpha}},\; {\rm sign}[B_{\boldsymbol{\alpha}}]\; E_{-\boldsymbol{\alpha}},\; \frac{2}{\boldsymbol{\alpha}^2}\,\boldsymbol{\alpha}\cdot\boldsymbol{h}\Big) \in g_{\mathbf{c}} \,. \tag{5.13}$$

Then, the one-soliton solution associated with $\boldsymbol{\alpha}$ is given by

$$h_{\boldsymbol{\alpha}}^{g} = \widehat{\iota}_{\boldsymbol{\alpha}}\big(h_{\boldsymbol{\alpha}}^{\rm CSG}\big) \in G \,, \tag{5.14}$$

where $\widehat{\iota}_{\boldsymbol{\alpha}}$ is the lift of $\iota_{\boldsymbol{\alpha}}$ to $SU(2)$. Actually, it is easy to show that $h_{\boldsymbol{\alpha}}^{g}$ satisfies eq. (2.9). Since $h_{\boldsymbol{\alpha}}^{g}$ is in the embedded $SU(2)$ subgroup of G, the left-hand-side of (2.9) is an element of the embedded $su(2)$ subalgebra of g, and only the components of $\Lambda_\pm$ along the Cartan generator $\boldsymbol{\alpha}\cdot\boldsymbol{h}$ contribute to the right-hand-side. Then, $\Lambda_\pm$ can be decomposed as

$$\Lambda_\pm \;=\; \pm\, i\, \boldsymbol{\lambda}_\pm\cdot\boldsymbol{h} \;=\; \pm\, i\, \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_\pm)}{\boldsymbol{\alpha}^2}\,\boldsymbol{\alpha}\cdot\boldsymbol{h} \;+\; \cdots \,, \tag{5.15}$$

where the dots indicate components which are orthogonal to $\boldsymbol{\alpha}\cdot\boldsymbol{h}$. Therefore, eq. (2.9) reduces in this case to

$$\partial_-\,\big(h_{\boldsymbol{\alpha}}^{g\,\dagger}\partial_+ h_{\boldsymbol{\alpha}}^{g}\big) \;=\; -\,\Big(\frac{m_{\boldsymbol{\alpha}}}{2}\Big)^2\,\Big[\big(\frac{1}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha}\cdot\boldsymbol{h}\big)\,,\; h_{\boldsymbol{\alpha}}^{g\,\dagger}\,\big(\frac{1}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha}\cdot\boldsymbol{h}\big)\,h_{\boldsymbol{\alpha}}^{g}\Big] \,, \tag{5.16}$$

which is just the embedding in g of the CSG equation satisfied by the $SU(2)$ field configuration $h_{\boldsymbol{\alpha}}^{\rm CSG}$.

Recall that the Killing form of g is normalized such that long roots have square length 2 and, hence, for non-simply laced algebras, the normalization of the Killing form is different in the original and the embedded $su(2)$ algebras. The relative normalization is given by

$$\langle \iota_{\boldsymbol{\alpha}}(\cdot)\,,\; \iota_{\boldsymbol{\alpha}}(\cdot)\rangle_g \;=\; \frac{2}{\boldsymbol{\alpha}^2}\,\langle\cdot\,,\;\cdot\rangle_{su(2)} \,, \tag{5.17}$$

as can be easily checked through the value of $\langle J_+ \,,\, J_- \rangle$.

Therefore, we conclude that the inequivalent solitons of the HSG theory associated with an arbitrary simple Lie group G are provided by the field configurations $h^g_{\boldsymbol{\alpha}} = \widehat{\imath}_{\boldsymbol{\alpha}}(h^{\rm CSG}_{\boldsymbol{\alpha}})$ with $\varphi \in (0,\pi)$ and $\boldsymbol{\alpha}$ a (positive) root of g. Notice that, in the particular case of simply-laced groups, eqs. (4.8) and (4.9) are just the matrix elements of $h^{\rm CSG}_{\boldsymbol{\alpha}}$ in the homogeneous vertex operator representation. Then, taking into account that the calculation of both the energy-momentum tensor and the action involves the Killing form of g, the mass and action of the soliton corresponding to φ are

$$M_{\boldsymbol{\alpha}}(\varphi) \;=\; \frac{1}{\pi\beta^2}\,\frac{2}{\boldsymbol{\alpha}^2}\, m_{\boldsymbol{\alpha}}\, |\sin\varphi| \tag{5.18}$$

$$S_{\boldsymbol{\alpha}}(\varphi) \;=\; \frac{2}{\beta^2}\,\frac{2}{\boldsymbol{\alpha}^2}\, \Big[\big|\, \pi\beta^2\, q(\varphi)\, \big| \;-\; |\tan\varphi| \Big]\,. \tag{5.19}$$

The value of the $U(1)^{\times r_g}$ conserved charge carried by this soliton can be obtained by means of the generalization of eq. (4.16),

$$h^{\rm CSG}_{\boldsymbol{\alpha}}(\pm\infty, t) \;=\; \begin{cases} I \stackrel{\widehat{\imath}_{\boldsymbol{\alpha}}}{\longmapsto} I & , \text{ if } \ \pm\sin\varphi > 0 \\ {\rm e}^{\,-2\,i\,\varphi\,\sigma_3} \stackrel{\widehat{\imath}_{\boldsymbol{\alpha}}}{\longmapsto} {\rm e}^{\,-2\,i\,\varphi\,\frac{2}{\boldsymbol{\alpha}^2}\,\boldsymbol{\alpha}\cdot\boldsymbol{h}} & , \text{ if } \ \pm\sin\varphi < 0 \ , \end{cases} \tag{5.20}$$

which leads to

$$\boldsymbol{Q}_{\boldsymbol{\alpha}}(\varphi) \;=\; q(\varphi)\, \frac{2}{\boldsymbol{\alpha}^2}\, \boldsymbol{\alpha} \quad \text{mod} \quad \frac{1}{\beta^2}\, \boldsymbol{\Lambda}^*_R\,, \tag{5.21}$$

where $q(\varphi)$ is specified by eq. (5.6), and $\boldsymbol{\Lambda}^*_R$ is the co-root lattice of g, *i.e.*, the lattice generated by the co-roots $\boldsymbol{\alpha}^* = (2/\boldsymbol{\alpha}^2)\,\boldsymbol{\alpha}$. Recall that, in their rest frame, solitons are given by time-dependent solutions that rotate in the internal $U(1)^{\times r_g}$ space with angular velocity

$$\omega_{\boldsymbol{\alpha}}(\varphi) \;=\; m_{\boldsymbol{\alpha}}\, \cos\varphi\,, \tag{5.22}$$

Then, each soliton of charge $\boldsymbol{Q}_{\boldsymbol{\alpha}}(\varphi)$ has a partner of charge $\boldsymbol{Q}_{\boldsymbol{\alpha}}(\pi-\varphi) = -\boldsymbol{Q}_{\boldsymbol{\alpha}}(\varphi)$ and the same mass that rotates with opposite angular velocity and, hence, that can be identified as its anti-particle.

The construction of soliton solutions presented in this Section is very similar to the construction of monopole solutions in theories with an adjoint Higgs field in the Prasad-Sommerfeld limit, *e.g.*, $N=2$ or $N=4$ supersymmetric gauge theories. In the latter case, monopole solutions are obtained by embeddings of the $SU(2)$ spherically symmetric 't Hooft-Polyakov monopole [18,19]. In a similar way, in the HSG theories, $\Lambda_\pm$ are analogous to the Higgs field vacuum expectation value and the role of the 't Hooft-Polyakov monopole is played by the $SU(2)$ CSG soliton.

In our construction we have only considered the regular embeddings of $SU(2)$ in G. However, a logical question is whether more general embeddings provide additional soliton solutions. Let us consider an arbitrary embedding $\iota : (J_\pm, J_0) \in su(2) \mapsto g$. Since the asymptotic behaviour of the embedded CSG soliton $h^\iota = \widehat{\iota}(h_{\boldsymbol{\alpha}}^{\rm CSG})$ is

$$h^\iota(\pm\infty, t) \;=\; \begin{cases} I & , \text{ if } \;\pm \sin\varphi > 0 \\ {\rm e}^{\,-2\,i\,\varphi\,\iota(J_0)} & , \text{ if } \;\pm\sin\varphi < 0\,, \end{cases} \tag{5.23}$$

we have to restrict ourselves to those embeddings such that $\iota(J_0)$ is in the Cartan subalgebra g_0^0 and, hence,

$$\iota(J_0) \;=\; \boldsymbol{f}\cdot\boldsymbol{h}\,, \quad \iota(J_+) \;=\; \sum_{\boldsymbol{\alpha}\in\Delta} c_{\boldsymbol{\alpha}}\, E_{\boldsymbol{\alpha}} \quad \text{and} \quad \iota(J_-) \;=\; {\rm sign}[B_{\boldsymbol{\alpha}}] \sum_{\boldsymbol{\alpha}\in\Delta} c_{\boldsymbol{\alpha}}\, E_{-\boldsymbol{\alpha}}\,, \tag{5.24}$$

where Δ is some set of positive roots of g. Then, h^ι can provide a solution of (2.9) only if $[\Lambda_\pm, \iota(su(2)_{\mathbf{c}})] = \iota(su(2)_{\mathbf{c}})$, which implies that the inner products $\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_\pm$ are equal for all the roots $\boldsymbol{\alpha}\in\Delta$. However, $\Lambda_\pm$ are regular elements of g_0^0 and, therefore, all this implies that $\boldsymbol{\alpha}\pm\boldsymbol{\beta}$ cannot be a root of g for any $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in Δ. In other words, h^ι provide a soliton solution only if $\iota(su(2))$ is the principal $su(2)$ subalgebra of some regular $A_1\oplus\cdots\oplus A_1$ subalgebra of g (for a nice review about $su(2)$ embeddings see [34] and references therein) and, in this case, h^ι is the product of the soliton solutions associated with all the roots in Δ:

$$h^\iota \;=\; \prod_{\boldsymbol{\alpha}\in\Delta} \widehat{\iota}_{\boldsymbol{\alpha}}\big(h_{\boldsymbol{\alpha}}^{\rm CSG}\big)\,. \tag{5.25}$$

Actually, if $\boldsymbol{\alpha}\pm\boldsymbol{\beta}$ are not roots of g it is straightforward to check that the product of $h_{\boldsymbol{\alpha}}^g = \widehat{\iota}_{\boldsymbol{\alpha}}\big(h_{\boldsymbol{\alpha}}^{\rm CSG}\big)$ and $h_{\boldsymbol{\beta}}^g$ is another solution of (2.9) without constraining the value of $\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_\pm$ and $\boldsymbol{\beta}\cdot\boldsymbol{\lambda}_\pm$, but this new solution has to be understood as a superposition of two non-interacting solitons. Therefore, we conclude that the solutions constructed by means of non-regular embeddings of $SU(2)$ in G correspond to the superposition of a certain number of non-interacting solitons and, hence, that they do not provide new soliton solutions different from those given by eq. (5.14).

6. Semi-classical spectrum

Having constructed the classical soliton solutions of the HSG theories, we will now address their quantization. At rest, soliton solutions provide explicit periodic time-dependent solutions and, therefore, we can apply the Bohr-Sommerfeld quantization rule: $S_{\boldsymbol{\alpha}}(\varphi) + M_{\boldsymbol{\alpha}}(\varphi)\, T_{\boldsymbol{\alpha}}(\varphi) \;=\; 2\pi\, n$, where $T_{\boldsymbol{\alpha}}(\varphi) \;=\; 2\pi/|\omega_{\boldsymbol{\alpha}}(\varphi)|$ is the period of the soliton

solution at rest and n is a positive integer. In our case, it is important to notice that the action (2.1) is multi-valued (modulo $2\pi/\beta^2$) and, consequently, the coupling constant β has to be quantized [21,22] [6]

$$\beta^2 = \frac{1}{k}\,, \quad k \in \mathbb{Z}^+\,. \tag{6.1}$$

Let us consider the semi-classical quantization of the soliton solutions associated with a positive root $\boldsymbol{\alpha}$ of g. Then, the results of the previous section lead to

$$S_{\boldsymbol{\alpha}}(\varphi) + M_{\boldsymbol{\alpha}}(\varphi)\, T_{\boldsymbol{\alpha}}(\varphi) = 2\pi\, \frac{2}{\boldsymbol{\alpha}^2}\, |q(\varphi)|\,, \tag{6.2}$$

and the quantization rule becomes

$$|q(\varphi)| = \frac{\boldsymbol{\alpha}^2}{2}\, n\,. \tag{6.3}$$

Classically, eq. (5.6) implies that $0 < |q(\varphi)| \leq k/2$. Moreover, the value of $q(\varphi)$ for the time independent solution obtained by setting $\varphi = \pi/2$ is not uniquely defined

$$\lim_{\varphi \to \frac{\pi}{2}^{\pm}} q(\varphi) = \mp \frac{k}{2}\,, \tag{6.4}$$

a property whose importance has been emphasized by Dorey and Hollowood in relation to the CSG theory [35]. This means that $q(\varphi) = \pm k/2$ actually represent the same classical soliton solution. This ambiguity can be avoided by identifying the values of $q(\varphi)$ modulo k $(= 1/\beta^2)$ [35], an identification that is naturally incorporated in our definition of the conserved charge $\boldsymbol{Q}_{\boldsymbol{\alpha}}(\varphi)$, eq. (5.21).

Taking into account all this, the semi-classical quantization of the soliton solutions associated with a positive root $\boldsymbol{\alpha}$ gives rise to $2k/\boldsymbol{\alpha}^2 - 1$ massive particles that can be labelled by an integer number

$$1 \leq n \leq \frac{2k}{\boldsymbol{\alpha}^2} - 1\,. \tag{6.5}$$

This number corresponds to $2q(\varphi)/\boldsymbol{\alpha}^2$ or $2(q(\varphi)+k)/\boldsymbol{\alpha}^2$, depending on whether $q(\varphi) > 0$ or $q(\varphi) < 0$, respectively. Then, the mass and charge carried by these particles are

$$M_{\boldsymbol{\alpha}}(n) = \frac{2k}{\pi\boldsymbol{\alpha}^2}\, m_{\boldsymbol{\alpha}}\, \left| \sin\left(\frac{\pi\boldsymbol{\alpha}^2}{2k}\, n\right) \right| \tag{6.6}$$

$$\boldsymbol{Q}_{\boldsymbol{\alpha}}(n) = n\,\boldsymbol{\alpha} \mod k\,\boldsymbol{\Lambda}_R^*\,, \quad n = 1, 2, \ldots, \frac{2k}{\boldsymbol{\alpha}^2} - 1\,, \tag{6.7}$$

[6] Quantum corrections are expected to induce a finite renormalization of $k \to k_R = k + N$, as it happens in the CSG theory [31]; however, the difference between k and k_R is negligible in the semi-classical $k \to \infty$ limit, and it will be ignored in the following.

and, now, the anti-particle of the particle n is labelled by $2k/\boldsymbol{\alpha}^2 - n$.

Notice that the $U(1)^{\times r_g}$ conserved charge, although continuous in the classical theory, should now more properly be though of as a discrete charge that takes values in $\boldsymbol{\Lambda}_R$ modulo $k\, \boldsymbol{\Lambda}_R^*$. Actually, it can be easily checked that the co-root lattice $\boldsymbol{\Lambda}_R^*$ is just the lattice spanned by the long roots of g. Therefore, $\boldsymbol{Q}_{\boldsymbol{\alpha}}(n)$ takes values in the global symmetry group of the theory of G-parafermions at level-k described by the gauged WZW action $S_{\rm WZW}[h, A_\pm]$ in eq. (2.1) [12], which points to the parafermionic character of the solitonic spectrum. We will comment about this in the last Section.

As it could have been anticipated, for a fixed root $\boldsymbol{\alpha}$ the spectrum is similar to the spectrum of the complex sine-Gordon theory [31,35], which itself resembles the spectrum of breather states in the ordinary sine-Gordon theory. However, notice that the number of particles resulting from the semi-classical quantization of the solitons associated with a root $\boldsymbol{\alpha}$ depends on its length. Thus, for long roots the number of resulting particles is always $k-1$, while for the short roots of B_n, C_n or F_4 is $2k-1$, and for those of G_2 it is $3k-1$. The lightest states in the spectrum of particles associated with a given root correspond to $n=1$ and $n=2k/\boldsymbol{\alpha}^2-1$, and they have charge $\boldsymbol{Q}_{\boldsymbol{\alpha}}(1) = -\,\boldsymbol{Q}_{\boldsymbol{\alpha}}(2k/\boldsymbol{\alpha}^2-1) = \boldsymbol{\alpha}$ and mass $M_{\boldsymbol{\alpha}}(1) = m_{\boldsymbol{\alpha}} + O(1/k^2)$. These quantum numbers are identical to those of the fundamental particle of the theory associated with the root $\boldsymbol{\alpha}$ (together with its anti-particle) and, therefore, we assume that these states describe the fundamental particle in the semi-classical limit. Moreover, $\boldsymbol{Q}_{\boldsymbol{\alpha}}(n) = n\,\boldsymbol{Q}_{\boldsymbol{\alpha}}(1)$ and

$$\frac{M_{\boldsymbol{\alpha}}(n)}{M_{\boldsymbol{\alpha}}(1)} \;=\; n \;-\; \frac{\pi^2\boldsymbol{\alpha}^4}{24k^2}\, n\,(n^2\;-\;1)\;+\;O(k^{-4})\,, \tag{6.8}$$

which suggests the obvious interpretation of the state labelled by n as a bound-state of n fundamental particles. Recall that all these identifications are possible because solitons do not carry topological charges, as it also happens in the CSG theory, and hence there is no topological distinction between the vacuum sector and the one-soliton sector.

A very peculiar property of these theories is that they exhibit unstable fundamental particles and solitons. To spell this out, let us start with the fundamental particles associated with three roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\alpha}+\boldsymbol{\beta}$. Then, it is easy to show that

$$\begin{aligned} m_{\boldsymbol{\alpha}+\boldsymbol{\beta}}^2 \;&=\; \big(m_{\boldsymbol{\alpha}} + m_{\boldsymbol{\beta}}\big)^2 \;+\; 4\, m_{\boldsymbol{\alpha}}\, m_{\boldsymbol{\beta}}\, \sinh^2\!\big(\sigma_{\boldsymbol{\alpha}} - \sigma_{\boldsymbol{\beta}}\big) \\ \text{where}\quad \sigma_{\boldsymbol{\alpha}} \;&=\; \frac{1}{2}\,\ln\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_+)}{(\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}_-)}\,, \end{aligned} \tag{6.9}$$

which implies that $m_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \geq m_{\boldsymbol{\alpha}} + m_{\boldsymbol{\beta}}$. Using this relation, one can deduce a lower bound for the mass of the fundamental particle associated with a root $\boldsymbol{\alpha} = \sum_{i=1}^{r_g} p_i \boldsymbol{\alpha}_i$,

$$m_{\boldsymbol{\alpha}} \geq \sum_{i=1}^{r_g} p_i\, m_{\boldsymbol{\alpha}_i}\;, \tag{6.10}$$

where $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{r_g}$ is a set of simple roots. To understand the implications of this bound for the soliton mass spectrum, let us consider

$$\begin{aligned}\sum_{i=1}^{r_g} M_{\boldsymbol{\alpha}_i}(n\, p_i) &= \sum_{i=1}^{r_g} \frac{2k}{\pi \boldsymbol{\alpha}_i^2}\, m_{\boldsymbol{\alpha}_i} \ \left| \sin\Bigl(\frac{\pi \boldsymbol{\alpha}_i^2}{2k}\, n\, p_i\Bigr) \right| \\ &= \frac{2k}{\pi \boldsymbol{\alpha}^2} \sum_{i=1}^{r_g} \frac{\boldsymbol{\alpha}^2}{\boldsymbol{\alpha}_i^2}\, m_{\boldsymbol{\alpha}_i} \ \left| \sin\Bigl(p_i\ \frac{\boldsymbol{\alpha}_i^2}{\boldsymbol{\alpha}^2}\ \frac{\pi \boldsymbol{\alpha}^2}{2k}\ n\Bigr) \right| ,\end{aligned} \tag{6.11}$$

for an arbitrary positive integer number n. Notice that $p_i = (2/\boldsymbol{\alpha}_i^2)\, \boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}$, where $\boldsymbol{\lambda}_i$ are the fundamental weights of g ($\boldsymbol{\alpha}_i \cdot \boldsymbol{\lambda}_j = (\boldsymbol{\alpha}_i^2/2)\, \delta_{i,j}$) and, hence, $p_i\, (\boldsymbol{\alpha}_i^2/\boldsymbol{\alpha}^2) = \boldsymbol{\lambda}_i \cdot \boldsymbol{\alpha}^*$ is a positive integer number. Then, using eq. (6.10) and the relation $|\sin(m\theta)| \leq m|\sin(\theta)|$ that is satisfied for $m \geq 1$, eq. (6.11) becomes

$$\begin{aligned}\sum_{i=1}^{r_g} M_{\boldsymbol{\alpha}_i}(n\, p_i) &\leq \frac{2k}{\pi \boldsymbol{\alpha}^2} \Bigl(\sum_{i=1}^{r_g} p_i\, m_{\boldsymbol{\alpha}_i}\Bigr) \ \left| \sin\Bigl(\frac{\pi \boldsymbol{\alpha}^2}{2k}\, n\Bigr) \right| \\ &\leq \frac{2k}{\pi \boldsymbol{\alpha}^2}\, m_{\boldsymbol{\alpha}} \ \left| \sin\Bigl(\frac{\pi \boldsymbol{\alpha}^2}{2k}\, n\Bigr) \right| = M_{\boldsymbol{\alpha}}(n)\, .\end{aligned} \tag{6.12}$$

This bound, together with

$$\boldsymbol{Q}_{\boldsymbol{\alpha}}(n) = \sum_{i=1}^{r_g} \boldsymbol{Q}_{\boldsymbol{\alpha}_i}(n\, p_i)\, , \tag{6.13}$$

shows that either the soliton particle labelled by $(\boldsymbol{\alpha}, n)$ is unstable and decays into solitons associated with the simple roots or, if the bound (6.12) is saturated, it is a bound-state at threshold. Similar phenomena occur for monopoles and dyons in $N = 2$ and $N = 4$ supersymmetric gauge theories [19].

Recall that unstable particles do not correspond to asymptotic scattering states and, hence, the only trace of them should be the existence of resonance poles in the S-matrix of the theory. As a consequence of the previous analysis, the scattering states of the HSG theories are expected to be described only by the solitons associated with the simple roots. These peculiarities will make the determination of the exact S-matrix of these theories and its confrontation with the results of semi-classical quantization extremely interesting.

7. Conclusions.

In this paper we have obtained the semi-classical spectrum of the HSG theories associated with arbitrary compact simple Lie groups. At the quantum level, these theories are massive perturbations of the level-k G-parafermions [12], where $1/k$ is the (quantized) coupling constant of the theory. There is a classical one-soliton solution associated with each

root $\boldsymbol{\alpha}$ of g, which can be constructed by embedding a CSG soliton in the $SU(2)$ subgroup of G generated by $E_{\boldsymbol{\alpha}}$, $E_{-\boldsymbol{\alpha}}$, and $[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}]$. For each root, the resulting semi-classical spectrum consists of a tower of $2k/\boldsymbol{\alpha}^2 - 1$ massive particles. The lightest one can be identified with the fundamental particle associated with the root $\boldsymbol{\alpha}$, which means that the full spectrum of the HSG theory is described by solitons. This supports the expectation that it should be possible to infer the form of the exact S-matrix by means of semi-classical methods, and makes possible to compare semi-classical results with standard perturbation theory in the large-k limit.

Unlike the kinks of the sine-Gordon equation, the solitons of the HSG theories do not have topological quantum numbers, but they carry a conserved vector Noether charge $\boldsymbol{Q}$ associated with a global $U(1)^{r_g}$ symmetry. After quantization, the charge is restricted to take values in $\boldsymbol{\Lambda}_R$ modulo $k\boldsymbol{\Lambda}_R^*$, *i.e.*, the root lattice of g, modulo k times the co-root lattice. Therefore, since $\boldsymbol{\Lambda}_R^*$ equals the long root lattice, the quantized charge carried by the solitons actually takes values in the discrete symmetry group of Gepner's theory of G-parafermions at level k [12].

The possibility of a semi-classical solitonic interpretation of parafermions was originally suggested by Bardakci *et al.* [36], but the relation between their results and ours is unclear. However, it is interesting to speculate on the identification of the HSG solitons with parafermionic fields. Following Gepner [12], and denoting by $\boldsymbol{\Lambda}_W$ the weight lattice of g, each field in the theory of level-k G-parafermions has a charge $(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ taking values in $\boldsymbol{\Lambda}_W \times \boldsymbol{\Lambda}_W$ modulo $k\boldsymbol{\Lambda}_R^* \times k\boldsymbol{\Lambda}_R^*$, and constrained by $\boldsymbol{\lambda} - \overline{\boldsymbol{\lambda}} \in \boldsymbol{\Lambda}_R$. On the other hand, for $h_0 = I$, the global symmetry transformations of the HSG theories, eq. (2.6), are of vector type, which makes the $U(1)^{r_g}$ charge carried by the $(\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}})$ field to be precisely $\boldsymbol{Q} = \boldsymbol{\lambda} - \overline{\boldsymbol{\lambda}}$. Therefore, the soliton particle labelled by $(\boldsymbol{\alpha}, n)$ should be related to fields whose charge is of the form $(n\boldsymbol{\alpha} + \boldsymbol{\lambda}, \boldsymbol{\lambda})$. In particular, the fundamental particle (anti-particle) associated with the root $\boldsymbol{\alpha}$ could well correspond to the 'generating parafermions' $\psi_{\boldsymbol{\alpha}}(z)$ $(\overline{\psi}_{\boldsymbol{\alpha}}(\overline{z}))$, although we do not have any definite argument to support a conjecture in that sense.

An important result is that some of the soliton particles are unstable, which means that the HSG theories actually capture important features of monopoles and dyons in $N = 2$ and $N = 4$ supersymmetric gauge theories in four dimensions [19]. To be precise, for $r_g \geq 2$, only the solitons associated with simple roots give rise to stable particles while the other ones are either unstable or bound-states at threshold. Therefore, in the S-matrix formalism, only stable soliton particles are expected to provide asymptotic scattering states, while the existence of unstable particles should manifest as resonance poles. A better understanding of these features requires a careful study of the scattering of soli-

tons which will be presented elsewhere.

Recently, an integrable perturbation of the $su(2)_k$ WZW model that also exhibits stable and unstable particles has been constructed by Brazhnikov [37]. Actually, one can show that this model is included in the clasification of [5] as a SSSG theory associated with the symmetric space $SU(3)/SO(3)$. This way, it can be generalized to provide a new series of integrable perturbation of WZW models that share most of the properties of the model constructed by Brazhnikov; in particular, the presence of stable and unstable particles in the spectrum [38]. These novel features make the HSG and SSSG theories quite unconventional because, up to our knowledge, the bootstrap program has been so far applied only to theories whose spectrum consists entirely of stable particles. All this provides extra motivation for the overall aim of finding their factorizable S-matrix.

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Appendix: Conventions about Lie and Kac-Moody algebras

Given a compact simple Lie algebra g and Lie group G, we shall use a subscript '$\mathbf{c}$' to denote their complexifications, *i.e.*, $g_{\mathbf{c}}$ and $G_{\mathbf{c}}$, respectively. Let us introduce a Chevalley basis for $g_{\mathbf{c}}$ which consists in Cartan generators $\boldsymbol{\mu} \cdot \boldsymbol{h}$ together with step operators $E_{\boldsymbol{\alpha}}$ for each root $\boldsymbol{\alpha}$, such that $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$ live in a $r_g = \text{rank}(g)$-dimensional vector space provided with an inner product normalized such that long roots have square length 2. These obey

$$[\boldsymbol{\mu} \cdot \boldsymbol{h}\,,\, E_{\boldsymbol{\alpha}}] \;=\; (\boldsymbol{\mu} \cdot \boldsymbol{\alpha})\, E_{\boldsymbol{\alpha}}\,, \quad \text{and} \quad [E_{\boldsymbol{\alpha}}\,,\, E_{-\boldsymbol{\alpha}}] \;=\; B_{\boldsymbol{\alpha}}\, \boldsymbol{\alpha} \cdot \boldsymbol{h}\,, \tag{A.1}$$

and the basis is completely specified by choosing $B_{\boldsymbol{\alpha}} = \langle E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}} \rangle$ to be either $+2/\boldsymbol{\alpha}^2$ or $-2/\boldsymbol{\alpha}^2$, and the sign is fixed once for all the roots. The first choice is standard in the context of matrix realizations of g, while the second is very convenient when considering vertex operator constructions.

Given $g_{\mathbf{c}}$, its compact real form g can be characterized through an automorphism θ of $g_{\mathbf{c}}$ that is 'anti-linear', meaning $\theta(a_1\varphi_1 + a_2\varphi_2) = a_1^*\,\theta(\varphi_1) + a_2^*\,\theta(\varphi_2)$ for any $a_1, a_2 \in \mathbb{C}$ and $\varphi_1, \varphi_2 \in g_{\mathbf{c}}$, and 'involutive', meaning $\theta^2 = I$. The relevant automorphism is defined through

$$\theta(E_{\pm\boldsymbol{\alpha}}) \;=\; -\operatorname{sign}[B_{\boldsymbol{\alpha}}]\, E_{\mp\boldsymbol{\alpha}} \equiv \,-\, E_{\pm\boldsymbol{\alpha}}^{\dagger} \;, \quad \theta(\boldsymbol{h}) \;=\; -\,\boldsymbol{h} \;\equiv\; -\,\boldsymbol{h}^{\dagger} \,. \tag{A.2}$$

Then, g consists of those elements of $g_{\mathbf{c}}$ fixed by the automorphism: $\theta(\varphi) = \varphi$.

The central extension of the loop algebra associated with $g_{\mathbf{c}}$ can be written as

$$\begin{aligned}
&\widehat{g} = \left\{u^{(m)} \;\middle|\; u \in g_{\mathbf{c}}\,,\; m \in \mathbb{Z}\right\} \,\oplus\, \mathbb{C}\, c\,, \\
&\left[u^{(m)}\,,\, v^{(n)}\right] \;=\; [u\,,\, v]^{(m+n)} \;+\; m\,\langle u\,,\, v\rangle\, c\,\delta_{m+n,0}\,, \\
&\left[d\,,\, u^{(m)}\right] \;=\; m\, u^{(m)}\,, \qquad \left[c\,,\, d\right] \;=\; \left[c\,,\, u^{(m)}\right] \;=\; 0\,.
\end{aligned} \tag{A.3}$$

Here, $\mathbb{C}c$ is the centre of $\widehat{g}$, and d is the derivation that induces the 'homogeneous gradation',

$$\begin{aligned}
\widehat{g} \;&=\; \bigoplus_{n\in\mathbb{Z}} \widehat{g}_{\,n}(\mathbf{s}_{\rm h})\,, \\
\widehat{g}_{\,n}(\mathbf{s}_{\rm h}) \;&=\; \left\{u^{(n)} \;\middle|\; u \in g_{\mathbf{c}}\right\} \,\oplus\, \left\{\mathbb{C}\, c\right\}\delta_{n,0}\,,
\end{aligned} \tag{A.4}$$

which can be labelled by the vector $\mathbf{s}_{\rm h} = (1, 0, \ldots, 0)$. [7] This way, $g_{\mathbf{c}}^{(1)} \;=\; \widehat{g} \oplus \mathbb{C}\, d$ is an 'untwisted affine Kac-Moody algebra' [39] whose Chevalley generators are

$$e_i^+ \;=\; \begin{cases} E_{\boldsymbol{\alpha}_i}^{(0)}\,, & \text{for } i = 1, \ldots, r_g\,, \\ E_{\boldsymbol{\alpha}_0}^{(1)}\,, & \text{for } i = 0\,, \end{cases} \qquad h_i \;=\; \frac{2}{\boldsymbol{\alpha}_i^2}\,\boldsymbol{\alpha}_i\cdot \boldsymbol{h}^{(0)} \;+\; c\,\delta_{i,0}\,. \tag{A.5}$$

In these equations, $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{r_g}$ are simple roots of $g_{\mathbf{c}}$, and $-\boldsymbol{\alpha}_0 = \sum_{i=1}^{r_g} k_i\,\boldsymbol{\alpha}_i$ is the highest root, which is always a long root and, hence, $\boldsymbol{\alpha}_0^2 = 2$.

Multi-soliton solutions are constructed by means of the 'integrable' highest-weight representations of $g_{\mathbf{c}}^{(1)}$, which can also be labelled by gradations $\mathbf{s} = (s_0, \ldots, s_{r_g})$ [17,39]. The highest-weight vector $|v_{\mathbf{s}}\rangle$ of the representation $L(\mathbf{s})$ satisfies

$$e_i^+\,|v_{\mathbf{s}}\rangle \;=\; \left(e_i^-\right)^{s_i+1}\,|v_{\mathbf{s}}\rangle \;=\; 0\,, \quad h_i\,|v_{\mathbf{s}}\rangle \;=\; s_i\,|v_{\mathbf{s}}\rangle\,, \tag{A.6}$$

[7] Recall that the different integer gradations of $\widehat{g}$ are labelled by sets $\mathbf{s} = (s_0, \ldots, s_{r_g})$ of non-negative integers that specify the grades of the Chevalley generators e_i^+ [17,39].

for all $i = 0, \ldots, r$. The eigenvalue of the centre c on $L(\mathbf{s})$ is known as the level k,

$$c\,|v_{\mathbf{s}}\rangle \;=\; \sum_{i=0}^{r} k_i^{\vee}\, h_i\, |v_{\mathbf{s}}\rangle \;=\; \left(\sum_{i=0}^{r} k_i^{\vee}\, s_i\right) |v_{\mathbf{s}}\rangle\;, \tag{A.7}$$

where $k_i^{\vee} = (\boldsymbol{\alpha}_i^2/2)\, k_i$ are the labels of the dual Dynkin diagram of $g_{\mathbf{c}}^{(1)}$. Therefore, in these representations, $k = \sum_{i=0}^{r} k_i^{\vee}\, s_i$, and the level is always a positive integer. Following [17], we will use the notation $L(i)$ and $|v_i\rangle$ for the fundamental representation and highest-weight vector corresponding to $\mathbf{s} = \mathbf{s}^{(i)}$ with $s_j^{(i)} = \delta_{i,j}$, whose level equals $k_i^{\vee}$. Then, the highest-weight vector of $L(\mathbf{s})$ can be decomposed as

$$|v_{\mathbf{s}}\rangle \;=\; \bigotimes_{i=0}^{r} \left\{|v_i\rangle^{\otimes s_i}\right\}. \tag{A.8}$$

For simply laced affine Kac-Moody algebras, all the fundamental integrable representations of level-one are isomorphic to the 'basic representation' $L(0)$, which can be realized by means of vertex operators acting on Fock spaces [40,27,28,29]. Then, the other fundamental integrable representations of level > 1 can be realized as submodules in the tensor product of several fundamental level-one representations. Moreover, the fundamental integrable representations of non-simply laced Kac-Moody algebras can be constructed from those of the simply laced algebras by folding them [41,42].

In Section 4 the soliton solutions of the HSG theory associated with a simply-laced Lie algebra g are constructed by means of the (level-one) 'homogeneous vertex operator construction' [27,28,29], which can be summarized as follows. First of all, if the representation is of level-one, the generators of the form $\boldsymbol{\mu}\cdot\boldsymbol{h}^{(n)}$ satisfy the (homogeneous) Heisenberg algebra

$$[\boldsymbol{\mu}\cdot\boldsymbol{h}^{(m)}\;,\;\boldsymbol{\gamma}\cdot\boldsymbol{h}^{(n)}] \;=\; m\,(\boldsymbol{\mu}\cdot\boldsymbol{\gamma})\,\delta_{m+n,0}\;, \tag{A.9}$$

which is equivalent to the commutation relations of r_g free bosonic fields. Thus, they can be represented as a set of oscillators $\boldsymbol{h}^{(n)} \mapsto \boldsymbol{a}_n$ acting on a Fock space. Let $|0\rangle$ be the vacuum vector for these oscillators and $\mathcal{F}$ the Fock space representation of the Heisenberg algebra spanned by the oscillators and the identity operator. Next, we identify $\boldsymbol{a}_0 = \boldsymbol{p}$ as a momentum operator and introduce a corresponding conjugate position operator $\boldsymbol{q}$ such that

$$[\boldsymbol{q}^j\;,\;\boldsymbol{p}^k] \;=\; i\,\delta^{j,k}\;,$$

and it commutes with all the other oscillators. Let W be the infinite-dimensional vector space spanned by vectors of the form $|\boldsymbol{\alpha}\rangle = \mathrm{e}^{\,i\,\boldsymbol{\alpha}\cdot\boldsymbol{q}}|0\rangle$ where $\boldsymbol{\alpha}$ is in the root lattice $\boldsymbol{\Lambda}_R$ of

g; obviously, $\boldsymbol{p}|\boldsymbol{\alpha}\rangle = \boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle$. For any root $\boldsymbol{\alpha}$ of g, let us define the vertex operator

$$\begin{aligned} V_{\boldsymbol{\alpha}}(z) &= \sum_{n\in\mathbb{Z}} z^{-n}\, V_{\boldsymbol{\alpha}}^{n} \\ &= z\, \exp\Big[\sum_{n>0} \frac{z^n}{n}\, \boldsymbol{\alpha}\cdot \boldsymbol{a}_{-n}\Big]\, \exp\Big[-\sum_{n>0} \frac{z^{-n}}{n}\, \boldsymbol{\alpha}\cdot \boldsymbol{a}_{n}\Big]\, \mathrm{e}^{\, i\, \boldsymbol{\alpha}\cdot \boldsymbol{q}}\, z^{\,\boldsymbol{\alpha}\cdot\boldsymbol{p}}\, C_{\boldsymbol{\alpha}}\, . \end{aligned} \tag{A.10}$$

In this definition, the $C_{\boldsymbol{\alpha}}$'s are a set of operators known as 'Klein factors' or 'cocycle operators', which act as

$$C_{\boldsymbol{\alpha}}\, |\boldsymbol{\beta}\rangle \;=\; \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})|\boldsymbol{\beta}\rangle\,, \quad \text{for all} \quad \boldsymbol{\beta}\in \boldsymbol{\Lambda}_R\,, \tag{A.11}$$

where $\epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})$ equals ± 1 and satisfies certain consistency conditions. Concerning $\epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})$, we will follow the conventions of ref. [29], which, in particular, imply that $\epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a 2-cocycle and that $\epsilon(\boldsymbol{\alpha},-\boldsymbol{\alpha}) = \epsilon(\boldsymbol{\alpha},\boldsymbol{\alpha}) = -1$.

Then, the homogenous vertex operator construction provides a realization of the basic representation $L(0)$ on the Hilbert space $\mathcal{F}\otimes W$ as follows:

$$\boldsymbol{h}^{(n)} \longmapsto \boldsymbol{a}_n\,, \qquad E_{\boldsymbol{\alpha}}^{(n)} \longmapsto V_{\boldsymbol{\alpha}}^{n}\,. \tag{A.12}$$

It is important to point out that, in this representation and with the conventions of [29], the generators $E_{\boldsymbol{\alpha}}^{n}$ satisfy the commutation relations

$$[E_{\boldsymbol{\alpha}}^{n}\,,\, E_{\boldsymbol{\beta}}^{m}] \;=\; \begin{cases} \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})\, E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}^{n+m} & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ is a root of } g \\ -\,\boldsymbol{\alpha}\cdot\boldsymbol{a}_{n+m} \;-\; m\,\delta_{m+n,0} & \text{if } \boldsymbol{\alpha}+\boldsymbol{\beta} = 0\,, \end{cases} \tag{A.13}$$

which implies the choice of $B_{\boldsymbol{\alpha}} = -1$ in eq. (A.1). The homogenous vertex operator construction also provides a realization of the other fundamental integrable representations of level-one. Let us consider a fundamental weight $\boldsymbol{\lambda}_i$ of g, *i.e.*, $\boldsymbol{\lambda}_i\cdot\boldsymbol{\alpha}_j = \delta_{i,j}$, such that $k_i^{\vee} = 1$ (recall that g is simply-laced). Then the fundamental representation $L(i)$ is realized through the same construction based on a different vacuum

$$|0\rangle \longmapsto |\boldsymbol{\lambda}_i\rangle \;=\; \mathrm{e}^{\, i\, \boldsymbol{q}\cdot\boldsymbol{\lambda}_i}\, |0\rangle\,, \tag{A.14}$$

together with a trivial modification of the Klein factors:

$$C_{\boldsymbol{\alpha}}\, |\boldsymbol{\lambda}_i + \boldsymbol{\beta}\rangle \;=\; \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})\, |\boldsymbol{\lambda}_i + \boldsymbol{\beta}\rangle\,, \quad \text{for all} \quad \boldsymbol{\beta}\in \boldsymbol{\Lambda}_R\,. \tag{A.15}$$

The product of two vertex operators can be normal-ordered such that

$$\begin{aligned} V_{\boldsymbol{\alpha}}(z)\, V_{\boldsymbol{\beta}}(w) \;=\; & z\, w\, (z-w)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}\, \exp\Big[\sum_{n>0} \frac{z^n\boldsymbol{\alpha} + w^n\boldsymbol{\beta}}{n}\cdot \boldsymbol{a}_{-n}\Big] \\ & \exp\Big[-\sum_{n>0} \frac{z^{-n}\boldsymbol{\alpha} + w^{-n}\boldsymbol{\beta}}{n}\cdot \boldsymbol{a}_{n}\Big]\, \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})\, \mathrm{e}^{\, i\, (\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{q}}\, C_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\, z^{\,\boldsymbol{\alpha}\cdot\boldsymbol{p}}\, w^{\,\boldsymbol{\beta}\cdot\boldsymbol{p}}\,, \end{aligned} \tag{A.16}$$

which implies the nilpotency of the vertex operator,

$$V_{\boldsymbol{\alpha}}(z)\, V_{\boldsymbol{\alpha}}(z) \ = 0\,, \tag{A.17}$$

and the relation $V_{\boldsymbol{\alpha}}(z) V_{\boldsymbol{\beta}}(w) = V_{\boldsymbol{\alpha}}(w) V_{\boldsymbol{\beta}}(z)$ for $z \neq w$.

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